Intermediate Mathematical Challenge

Thursday 5th February 2015

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Intermediate Mathematical Challenge (IMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the Intermediate Mathematical Olympiad and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	E	E	C	B	D	E	B	B	C	D	D	A	C	C	D	D	A	A	C	D	B	B	E	D

1. What is the value of $1 - 0.2 + 0.03 - 0.004$?

A 0.826 B 0.834 C 0.926 D 1.226 E 1.234

SOLUTION **A**

We have $1 - 0.2 = 0.8$ and so $1 - 0.2 + 0.03 = 0.8 + 0.03 = 0.83$, and therefore $1 - 0.2 + 0.03 - 0.004 = 0.83 - 0.004 = 0.826$.

Note that in the context of the IMC where we just need to decide which of the options is correct, it isn't really necessary to do this exact arithmetic. Looking at the terms making up the sum it is evident that the answer is just greater than 0.8, and the presence of the term -0.004 shows that the digit in the thousandths column is 6. So A is the correct option.

FOR INVESTIGATION

1.1 What is the value of $1-0.2+0.03-0.004+0.000\,5-0.000\,06+0.000\,007-0.000\,000\,8+0.000\,000\,09$?

2. Last year, Australian Suzy Walsham won the annual women's race up the 1576 steps of the Empire State Building in New York for a record fifth time. Her winning time was 11 minutes 57 seconds.

Approximately how many steps did she climb per minute?

A 13 B 20 C 80 D 100 E 130

SOLUTION **E**

We see that 11 minutes 53 seconds is just under 12 minutes, and 1576 is just under 1600. So the number of steps per minute is approximately

$$\frac{1600}{12} = \frac{400}{3} \sim 133.$$

We conclude that, of the given options, 130 is the best approximation. [There are other methods.]

3. What is a half of a third, plus a third of a quarter, plus a quarter of a fifth?

A $\frac{1}{1440}$ B $\frac{3}{38}$ C $\frac{1}{30}$ D $\frac{1}{3}$ E $\frac{3}{10}$

SOLUTION **E**

A half of a third is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, a third of a quarter is $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$, and a quarter of a fifth is $\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$.

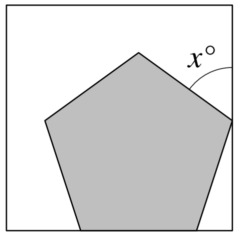
It follows that the required answer is

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{10+5+3}{60} = \frac{18}{60} = \frac{3}{10}.$$

4. The diagram shows a regular pentagon inside a square.

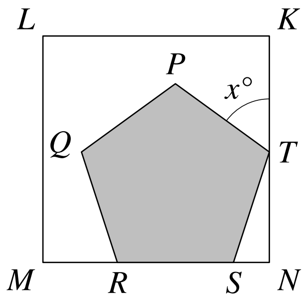
What is the value of x?

A 48 B 51 C 54 D 60 E 72



SOLUTION **C**

METHOD 1



Let the vertices of the square be labelled K, L, M, N and those of the regular pentagon P, Q, R, S, T, as shown in the diagram.

We use the fact that the interior angles of a regular pentagon are each 108°. You are asked to show this in Problem 4.1. It follows that $\angle RST = \angle STP = 108°$. Because it is the angle of a square, $\angle SNT = 90°$. By the Exterior Angle Theorem, $\angle RST = \angle SNT + \angle NTS$. Therefore, $108° = 90° + \angle NTS$. It follows that $\angle NTS = 108° - 90° = 18°$.

Finally, because the angles at T on the line KN have sum 180°,

$$\angle NTS + \angle STP + \angle PTK = 180°,$$

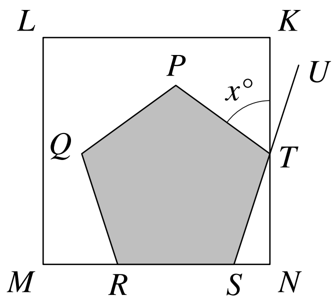
that is,

$$18° + 108° + x° = 180°.$$

We conclude that

$$x = 180 - 18 - 108 = 54.$$

METHOD 2



We extend ST to the point U as shown.

The external angle of a regular pentagon is 72°. Hence $\angle TSN = \angle PTU = 72°$. Therefore, from the right-angled triangle STN, we can deduce that $\angle STN = 90° - 72° = 18°$. So the vertically opposite angle $\angle KTU$ is also 18°. It follows that $\angle PTK = \angle PTU - \angle KTU = 72° - 18° = 54°$. Therefore $x = 54$.

FOR INVESTIGATION

4.1 Show that the interior angles of a regular pentagon are each 108° and that the exterior angles are each 72°. [Note that it might be helpful to look at Problem 9.1, below.]

4.2 Find a formula for the interior angles of a regular polygon with n sides. [Again, Problem 9.1, below, is relevant.]

4.3 What would x be in the case where the regular pentagon is replaced by
(a) a regular hexagon? (b) a regular heptagon? (c) a regular octagon?

4.4 What happens in the case where the regular pentagon is replaced by a regular nonagon? [Note that a *nonagon* is a polygon with 9 sides.]

5. Which of the following numbers is not a square?

A 1^6 B 2^5 C 3^4 D 4^3 E 5^2

Solution **B**

We have $2^5 = 32$ which is not a square.

In the context of the IMC we can assume that only one option is correct and that it is sufficient just to find it. However, for a complete solution we need to check that none of the other options is a square. Note that we can show this without evaluating any of them, as follows. We have $1^6 = (1^3)^2$, $3^4 = (3^2)^2$, $4^3 = (2^2)^3 = 2^6 = (2^3)^2$, and so these are all squares, and it is immediate that 5^2 is a square.

For investigation

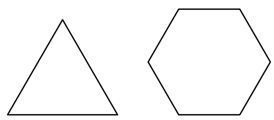
5.1 Which of the following are squares: (a) 2015^{2015}, (b) 49^{2015}, (c) 2015^{2016}?

5.2 Suppose that the positive integer m is a square. For which positive integers n is m^n a square?

5.3 Suppose that the positive integer m is not a square. For which positive integers n is m^n a square?

6. The equilateral triangle and regular hexagon shown have perimeters of the same length.

What is the ratio of the area of the triangle to the area of the hexagon?

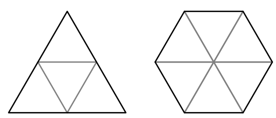


A 5 : 6 B 4 : 5 C 3 : 4 D 2 : 3 E 1 : 1

Solution **D**

We can assume that we have chosen units so that each side of the regular hexagon has length 1. It follows that the length of the perimeter of this hexagon is 6. Therefore the length of the perimeter of the equilateral triangle is 6. Hence each of its sides has length 2.

It follows that we can divide the equilateral triangle into 4 smaller equilateral triangles each with side length 1 as shown. The hexagon may be divided into 6 equilateral triangles with side length 1 as shown.



It follows that the ratio of their areas is 4 : 6, that is, 2 : 3.

For investigation

6.1 What is the area of an equilateral triangle with side length a?

6.2 What is the area of an equilateral triangle with a perimeter of length $6a$?

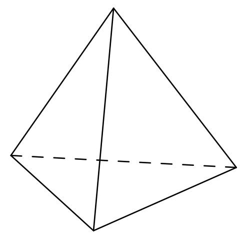
6.3 What is the area of a regular hexagon with a perimeter of length $6a$?

6.4 Check that your answers to 6.2 and 6.3 are in the ratio 2 : 3.

7. A tetrahedron is a solid figure which has four faces, all of which are triangles.

What is the product of the number of edges and the number of vertices of the tetrahedron?

A 8 B 10 C 12 D 18 E 24



SOLUTION **E**

We can think of a tetrahedron as a pyramid with a triangular base.

The base has 3 vertices and there is 1 more vertex, called the *apex*, at the peak of the pyramid. So a tetrahedron has 4 vertices.

The base has 3 edges, and there are 3 more edges joining the vertices of the base to the apex. Therefore a tetrahedron has $3 + 3 = 6$ edges.

Finally, we observe that $4 \times 6 = 24$.

8. How many two-digit squares differ by 1 from a multiple of 10?

A 1 B 2 C 3 D 4 E 5

SOLUTION **B**

The two-digit squares are 16, 25, 36, 49, 64 and 81. Of these, just $49 = 50 - 1$ and $81 = 80 + 1$ differ by 1 from a multiple of 10. So there are 2 such two-digit squares.

FOR INVESTIGATION

8.1 How many three-digit squares differ by 1 from a multiple of 10?

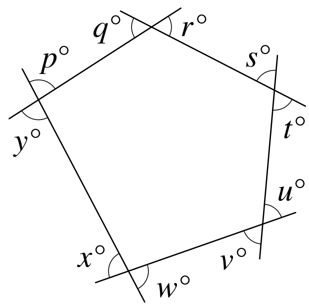
8.2 How many four-digit squares differ by 1 from a multiple of 10?

8.3 How many ten-digit squares differ by 1 from a multiple of 10?

8.4 Let n be a positive integer. How many squares with $2n - 1$ or $2n$ digits differ by 1 from a multiple of 10?

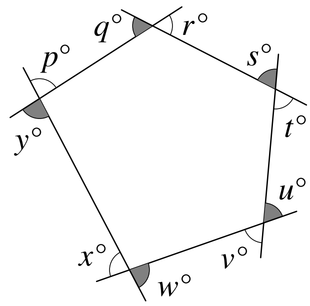
9. What is the value of $p + q + r + s + t + u + v + w + x + y$ in the diagram?

A 540 B 720 C 900 D 1080 E 1440



Solution **B**

We use the fact that for every polygon the sum of its exterior angles is 360°. See the note below for how this statement should be interpreted.



The shaded angles in the diagram on the right form one set of external angles. Therefore

$$q + s + u + w + y = 360.$$

The other marked angles form another set of external angles and therefore

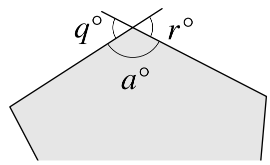
$$r + t + v + x + p = 360.$$

[Alternatively, we could argue that the angles marked $q°$ and $r°$ are vertically opposite and therefore $q = r$, and, similarly, $s = t$, $u = v$, $w = x$ and $p = y$, and therefore $r + t + v + x + p = q + s + u + w + y = 360$.]

It follows that

$$\begin{aligned} p + q + r + s + t + u + v + w + x + y + z &= (q + s + u + w + y) + (r + t + v + x + p) \\ &= 360 + 360 \\ &= 720. \end{aligned}$$

Note



The figure on the right shows part of the pentagon. One of the interior angles of this pentagon has been marked $a°$. There are two *exterior angles* associated with this interior angle, those marked $q°$ and $r°$. When we talk about the *sum of the exterior angles* we mean the sum obtained by choosing, for each vertex, just one of the two exterior angles associated with it. Because these are vertically opposite angles, they are equal, and so it does not matter which one we choose. For example, with the pentagon of this question both $q + s + u + w + y$ and $r + t + v + x + p$ give the sum of the exterior angles.

For investigation

9.1 Show that for every polygon the sum of its exterior angles is 360°.

10. What is the remainder when $2^2 \times 3^3 \times 5^5 \times 7^7$ is divided by 8?

A 2 B 3 C 4 D 5 E 7

SOLUTION **C**

First, we note that $3^3 \times 5^5 \times 7^7$ is a positive odd integer greater than 1 and so is equal to $2n + 1$, for some positive integer n.

It follows that

$$\begin{aligned} 2^2 \times 3^3 \times 5^5 \times 7^7 &= 4 \times (3^3 \times 5^5 \times 7^7) \\ &= 4(2n + 1) \\ &= 8n + 4. \end{aligned}$$

Therefore when $2^2 \times 3^3 \times 5^5 \times 7^7$ is divided by 8, the quotient is n and the remainder is 4.

FOR INVESTIGATION

10.1 What is the remainder when $2^2 \times 3^3 \times 5^5 \times 7^7 \times 11^{11}$ is divided by 8?

10.2 What is the remainder when $2^2 \times 3^3 \times 5^5 \times 7^7$ is divided by 16?

11. Three different positive integers have a mean of 7.

What is the largest positive integer that could be one of them?

A 15 B 16 C 17 D 18 E 19

SOLUTION **D**

Because the three integers have mean 7, their sum is $3 \times 7 = 21$. For the largest of the three positive integers to be as large as possible the other two need to be as small as possible. Since they are different positive integers, the smallest they can be is 1 and 2. The largest of the three integers is then $21 - (1 + 2) = 18$. So 18 is the largest any of them could be.

FOR INVESTIGATION

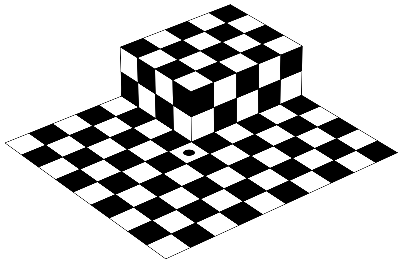
11.1 Five different positive integers have a mean of 10. What is the largest positive integer that could be one of them?

11.2 A set of different positive integers has mean 10. The largest number in the set is 50. What is the maximum number of positive integers that there could be in the set?

12. An ant is on the square marked with a black dot. The ant moves across an edge from one square to an adjacent square four times and then stops.

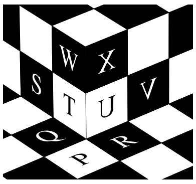
How many of the possible finishing squares are black?

A 0 B 2 C 4 D 6 E 8



Solution **D**

The figure shows the part of the layout that is relevant. Some of the squares have been labelled so that we may refer to them. A movement of the ant will be indicated by the sequence of labels of the four squares, in order, that the ant travels to after leaving the square labelled P. For example, QTUV is one such sequence. In this example the ant ends up on the black square V.



The ant starts on a white square. If at each move the colour of the square it is on changes, after four moves it will end up on a white square. So the only way the ant can end up on a black square is if on one of its moves the colour of the square does not change. That is, during its sequence of moves it either moves from T to U, or from U to T, or from W to X, or from X to W.

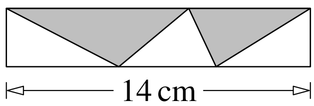
We therefore see that the only movements which result in the ant ending up on a black square are

QTUR, QTUV, QTUX, QTWX, RUTQ, RUTS, RUTW and RUXW.

From the final letters of these sequences we see that there are exactly 6 black squares that the ant can end up on after four moves, namely, Q, R, S, V, W and X.

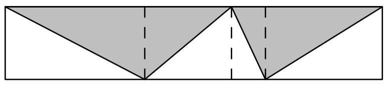
13. What is the area of the shaded region in the rectangle?

A $21\,\text{cm}^2$ B $22\,\text{cm}^2$ C $23\,\text{cm}^2$
D $24\,\text{cm}^2$ E more information needed



Solution **A**

We divide the rectangle into four smaller rectangles by the vertical dashed lines as shown in the figure. It is then evident that the shaded area is the sum of exactly half the area of each of the smaller rectangles.



Therefore the total shaded area is half the total area of the original rectangle. So the shaded area is $\frac{1}{2}(3 \times 14)\,\text{cm}^2 = 21\,\text{cm}^2$.

We could also solve this question algebraically. We let the top edges of the two shaded triangles have lengths a cm and b cm, respectively, so that $a + b = 14$. The two shaded triangles have areas $\frac{1}{2}(a \times 3)\,\text{cm}^2 = \frac{3}{2}a\,\text{cm}^2$ and $\frac{1}{2}(b \times 3)\,\text{cm}^2 = \frac{3}{2}b\,\text{cm}^2$, respectively. Hence their total area is

$$\left(\tfrac{3}{2}a + \tfrac{3}{2}b\right)\text{cm}^2 = \tfrac{3}{2}(a+b)\,\text{cm}^2 = \tfrac{3}{2} \times 14\,\text{cm}^2 = 21\,\text{cm}^2.$$

14. In a sequence, each term after the first two terms is the mean of all the terms which come before that term. The first term is 8 and the tenth term is 26.

What is the second term?

A 17 B 18 C 44 D 52 E 68

Solution **C**

Here it is actually a little easier to see what is going on if we think about the general case of a sequence in which every term after the second is the mean of all the terms that come before it.

Consider a sequence generated by this rule whose first two terms are a and b. The mean of these terms is $\frac{a+b}{2}$, and so the first three terms are

$$a,\ b,\ \frac{a+b}{2}.$$

The sum of these three terms is the same as the sum of the three terms

$$\frac{a+b}{2},\ \frac{a+b}{2},\ \frac{a+b}{2}$$

and it follows that the mean of the first three terms is also $\frac{a+b}{2}$. So the first four terms of the sequence are

$$a,\ b,\ \frac{a+b}{2},\ \frac{a+b}{2}.$$

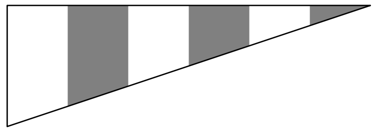
These have the same sum as the four terms

$$\frac{a+b}{2},\ \frac{a+b}{2},\ \frac{a+b}{2},\ \frac{a+b}{2}.$$

It follows that the mean of the first four terms is also $\frac{a+b}{2}$ and that therefore this is the fifth term of the sequence, and so on. We thus see that each term of the sequence after the first two is equal to the mean of the first two terms.

In the sequence we are given in the question the first term is 8 and the tenth term is 26. It follows that 26 is the mean of the first two terms. Hence their sum is $2 \times 26 = 52$. Since the first term is 8, we deduce that the second term is $52 - 8 = 44$.

15. A flag is in the shape of a right-angled triangle, as shown, with the horizontal and vertical sides being of length 72 cm and 24 cm respectively. The flag is divided into 6 vertical stripes of equal width.

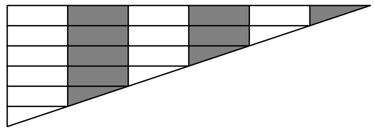


What, in cm^2, is the difference between the areas of any two adjacent stripes?

A 96 B 72 C 48 D 32 E 24

Solution **C**

The lines we have added in the figure on the right divide the flag into 15 congruent rectangles and 6 congruent triangles. Let the area of each of the 15 rectangles be $x\,\text{cm}^2$. The area of each of the 6 congruent triangles is half that of the rectangles, that is, $\frac{1}{2}x\,\text{cm}^2$.



The difference between the area of the adjacent vertical stripes of the flag is the area of one of the rectangles.

From its dimensions, we see that the area of the flag is $\frac{1}{2}(72 \times 24)\,\text{cm}^2 = (36 \times 24)\,\text{cm}^2$. It is made up of 15 rectangles, each with area $x\,\text{cm}^2$, and 6 triangles, each with area $\frac{1}{2}x\,\text{cm}^2$.

Therefore

$$15x + 6(\tfrac{1}{2}x) = 36 \times 24,$$

that is,

$$18x = 36 \times 24,$$

and it follows that

$$x = \frac{36 \times 24}{18} = 2 \times 24 = 48.$$

Hence the difference between the area of adjacent stripes of the flag is $48\,\text{cm}^2$.

For investigation

15.1 If the flag were divided into 8 vertical stripes of equal width, what would be the difference between the areas of any two adjacent stripes?

16. You are asked to choose two positive integers, m and n with $m > n$, so that as many as possible of the expressions $m + n$, $m - n$, $m \times n$ and $m \div n$ have values that are prime.

When you do this correctly, how many of these four expressions have values that are prime?

A 0 B 1 C 2 D 3 E 4

Solution **D**

With $m = 2$ and $n = 1$, we have $m + n = 3$, $m - n = 1$, $m \times n = 2$ and $m \div n = 2$. So it is possible for three of the expressions to have values that are prime. [Note that 1 is *not* a prime, so in this case $m - n$ is not prime. See the *Note* below for a discussion of this point.]

We now show that this is the best that can be achieved by showing that it is not possible for all four of the expressions to have values that are primes.

Since m and n are positive integers with $m > n$, we have $m > 1$. Therefore $m \times n$ is prime if, and only if, m is prime and $n = 1$. Then $m + n = m + 1$ and $m - n = m - 1$. Therefore, if all four expressions are primes, $m - 1$, m and $m + 1$ would be three consecutive primes. However a trio of consecutive positive integers that are all primes does not exist (see Problem 16.3). We conclude that not all four of the expressions can have values that are primes.

For investigation

16.1 For which values of m and n, other than $m = 2$ and $n = 1$, do three of the expressions have values that are primes?

16.2 In our example with $m = 2$ and $n = 1$ three of the expressions have values that are primes, but these values include just two different primes, namely, 2 and 3. Is it possible for three of the expressions $m + n$, $m - n$, $m \times n$, $m \div n$ to take three different prime values?

16.3 Prove there is no positive integer m such that $m - 1$, m and $m + 1$ are all primes.

Note

The standard definition of a *prime number* is that it is a positive integer which has exactly two factors. You can see that, according to this definition, 1 is not a prime number.

It is only a convention that makes us now adopt a definition according to which 1 is not a prime. (Earlier writers were not consistent about this.) However, it is a sensible convention. One reason, among many, for this is that it enables us to state *The Fundamental Theorem of Arithmetic* in a simple way. This theorem says that "each positive integer greater than 1 can be expressed as a product of prime numbers in just one way".

Of course, we can often write the prime factors of a given integer in more than one order. For example, 12 may be expressed as $2 \times 2 \times 3$, $2 \times 3 \times 2$ and $3 \times 2 \times 2$, but in each case the prime factor 2 occurs twice, and the prime factor 3 occurs once. This is what the Fundamental Theorem means by "in just one way". We would need to state this theorem in a more complicated way if we regarded 1 as a prime number. (If 1 counted as a prime then, for example, 12 would have infinitely many different factorizations into primes: $2 \times 2 \times 3$, $1 \times 2 \times 2 \times 3$, $1 \times 1 \times 2 \times 2 \times 3$, and so on, with any number of 1s.)

17. The football shown is made by sewing together 12 black pentagonal panels and 20 white hexagonal panels. There is a join wherever two panels meet along an edge.

How many joins are there?

A 20 B 32 C 60 D 90 E 180

Solution **D**

Each of the 12 pentagons has 5 edges, and each of the 20 hexagons has 6 edges. So, altogether, the panels have $12\times5+20\times6 = 180$ edges. Each join connects 2 edges, So there are $\frac{1}{2}(180) = 90$ joins.

Note

There is a good discussion of the shapes of footballs in the book *Beating the odds: the hidden mathematics of sport* by Rob Eastaway and John Haigh. The following problems are based on this book.

For investigation

17.1 When a football is inflated, the panels become curved, so a football is not a polyhedron with flat faces. However a football can be thought of as an inflated polyhedron. The football illustrated in this question is obtained by inflating a polyhedron made up of 12 regular pentagons and 20 regular hexagons.

There are other possibilities.

(a) An inflated *dodecahedron* made up of 12 panels, all regular pentagons.

(b) An inflated *icosahedron* made up of 20 panels, all equilateral triangles.

(c) An inflated *rhombicosidodecahedron* made up of 20 equilateral triangles, 30 squares and 12 regular pentagons.

For each of these shapes, work out how many joins would be needed to make it.

17.2 The football in the question is obtained by inflating the polyhedron which is called a *truncated icosahedron*. Why does it have this name?

17.3 Why is the design in the question preferred by some manufacturers of footballs?

18. The total weight of a box, 20 plates and 30 cups is 4.8 kg. The total weight of the box, 40 plates and 50 cups is 8.4 kg.

What is the total weight of the box, 10 plates and 20 cups?

A 3 kg B 3.2 kg C 3.6 kg D 4 kg E 4.2 kg

SOLUTION **A**

The difference between the box, 20 plates, 30 cups and the box, 40 plates, 50 cups, is 20 plates and 20 cups. From the information in the question we see that the extra weight of 20 plates and 20 cups is $8.4\,\text{kg} - 4.8\,\text{kg} = 3.6\,\text{kg}$. Therefore the weight of 10 plates and 10 cups is $\frac{1}{2}(3.6)\,\text{kg} = 1.8\,\text{kg}$.

If we remove 10 plates and 10 cups from the box, 20 plates and 30 cups we end up with the box, 10 plates and 20 cups. Removing the 10 plates and 10 cups reduces the original weight of 4.8 kg by 1.8 kg. Hence the weight of the box, 10 plates and 20 cups is $4.8\,\text{kg} - 1.8\,\text{kg} = 3\,\text{kg}$.

FOR INVESTIGATION

18.1 This problem could also have been solved using the language of algebra.

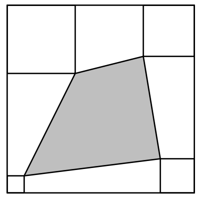
We let x, y and z be the weights, in kg, of the box, 1 plate and 1 cup, respectively.

(a) Write down the two equations that correspond to the information given in the question.

(b) Use these equations to find the value of $x + 10y + 20z$ which gives the weight, in kg, of the box, 10 plates and 20 cups.

(c) Note that, as we have only two equations relating the three unknowns, x, y and z, it is not possible to solve them to obtain definite values for each of the unknowns. However, it is possible to use these equations to find the value of certain combinations of x, y and z. For example, we have already found the value of $x + 10y + 20z$. Investigate the values of α, β and γ for which it is possible to calculate the value of $\alpha x + \beta y + \gamma z$.

19. The figure shows four smaller squares in the corners of a large square. The smaller squares have sides of length 1 cm, 2 cm, 3 cm and 4 cm (in anticlockwise order) and the sides of the large square have length 11 cm.



What is the area of the shaded quadrilateral?

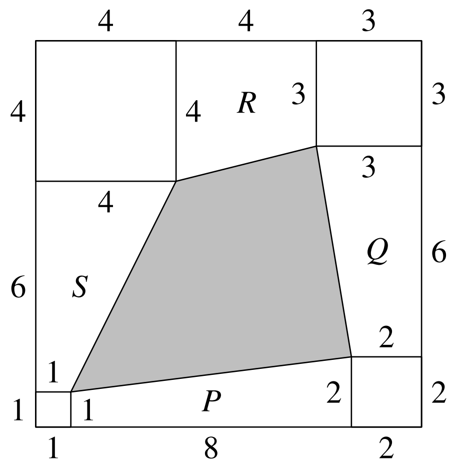
A $35\,\text{cm}^2$ B $36\,\text{cm}^2$ C $37\,\text{cm}^2$ D $38\,\text{cm}^2$ E $39\,\text{cm}^2$

SOLUTION **A**

We give two methods for answering this question. The first is a routine calculation. The second method is more elegant and involves fewer calculations.

METHOD 1

We work out the area that is not shaded and subtract this area from that of the large square.



The area that is not shaded is made up of the four corner squares and four trapeziums.

The total area of the squares is, in cm^2,

$$1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30.$$

We now use the fact that the area of a trapezium is given by the formula

$$\tfrac{1}{2}(a + b)h,$$

where a and b are the lengths of the parallel sides, and h is their distance apart. You are asked to prove this in Problem 19.1.

The values of a, b and h for the trapeziums P, Q, R and S, as labelled in the diagram, can be calculated from the side lengths of the squares, and are shown in the diagram. We therefore have that

$$\begin{aligned} \text{area of } P &= \tfrac{1}{2}(1 + 2) \times 8 = 12,\\ \text{area of } Q &= \tfrac{1}{2}(2 + 3) \times 6 = 15,\\ \text{area of } R &= \tfrac{1}{2}(3 + 4) \times 4 = 14,\\ \text{and} \quad \text{area of } S &= \tfrac{1}{2}(4 + 1) \times 6 = 15. \end{aligned}$$

It follows that the total area of the trapeziums is

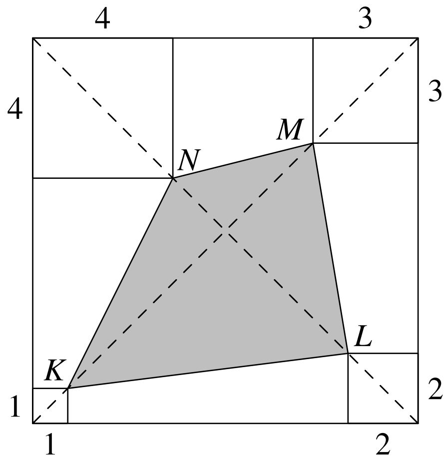
$$12 + 15 + 14 + 15 = 56.$$

The large square has area $11^2\,\text{cm}^2 = 121\,\text{cm}^2$. Therefore the area of the shaded quadrilateral is, in cm^2,

$$121 - 30 - 56 = 35.$$

Method 2

We label the vertices of the shaded quadrilateral K, L, M and N, as shown.



Because K, L, M and N are vertices of the small squares it can be seen that they lie on the diagonals of the large square. The diagonals of a square meet at right angles. Therefore KM is at right angles to LN.

As the diagonals of the quadrilateral $KLMN$ are at right angles, its area is half the product of the lengths of its diagonals. We ask you to prove this in Problem 19.2.

Thus, to complete the answer all we need do is find the lengths of KM and LN.

In Problem 19.3 we ask you to show that a square with side length s has diagonals of length $s\sqrt{2}$.

It follows from this that the diagonals of the large square have length $11\sqrt{2}$ cm, and that the diagonals of the small squares have lengths $\sqrt{2}$ cm, $2\sqrt{2}$ cm, $3\sqrt{2}$ cm and $4\sqrt{2}$ cm. Therefore KM has length $(11-1-3)\sqrt{2}\,\text{cm} = 7\sqrt{2}\,\text{cm}$ and LN has length $(11-2-4)\sqrt{2}\,\text{cm} = 5\sqrt{2}\,\text{cm}$.

We can now deduce that the area of the shaded quadilateral $KLMN$ is

$$\tfrac{1}{2}(7\sqrt{2}\times 5\sqrt{2})\,\text{cm}^2 = 35\,\text{cm}^2.$$

For investigation

19.1 Show that the area of a trapezium is $\frac{1}{2}(a+b)h$, where a and b are the lengths of the parallel sides and h is their distance apart.

19.2 Show that if the diagonals of a quadralateral meet at right angles, then the area of the quadrilateral is $\frac{1}{2}de$, where d and e are the lengths of the diagonals.

19.3 Show that if the side length of a square is s, then its diagonals have length $s\sqrt{2}$.

20. A voucher code is made up of four characters. The first is a letter: V, X or P. The second and third are different digits. The fourth is the units digit of the sum of the second and third digits.

How many different voucher codes like this are there?

A 180 B 243 C 270 D 300 E 2700

Solution **C**

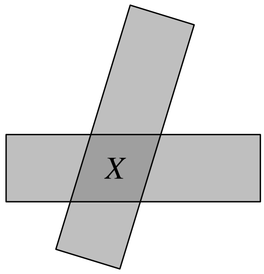
There are 3 choices for the first character, namely V, X or P. There are 10 choices for the second character, namely one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. Because the second and third characters are different digits, the third character can be any one of the 10 digits, other than the second character. Therefore there are 9 choices for the third character. There is just 1 choice for the fourth character as this is the units digit of the sum of the second and third digits.

It follows that the total number of voucher codes is $3\times 10\times 9\times 1 = 270$.

21. A rectangle is placed obliquely on top of an identical rectangle, as shown. The area X of the overlapping region (shaded more darkly) is one eighth of the total shaded area.

What fraction of the area of one rectangle is X?

A $\frac{1}{3}$ B $\frac{2}{7}$ C $\frac{1}{4}$ D $\frac{2}{9}$ E $\frac{1}{5}$



Solution **D**

The rectangles are identical. Therefore they have the same area. Let this area be Y.

It follows that the area of each rectangle that is not part of the overlap is $Y - X$. Hence the total shaded area, made up of the two areas of the rectangles which are not part of the overlap, together with the overlap, is

$$2(Y - X) + X = 2Y - X.$$

We are given that the area of the overlap is one eighth of the total shaded area. Therefore

$$\frac{X}{2Y - X} = \frac{1}{8}.$$

This equation may be rearranged to give

$$8X = 2Y - X,$$

from which it follows that

$$9X = 2Y,$$

and therefore that

$$\frac{X}{Y} = \frac{2}{9}.$$

For investigation

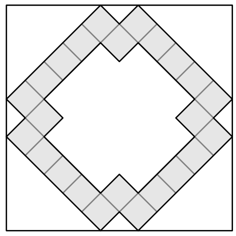
21.1 If the area X is one tenth of the total shaded area, what fraction of the area of one rectangle is X?

21.2 If the area X is one quarter of the area of one rectangle, what fraction is X of the total shaded area?

22. The diagram shows a shaded region inside a large square. The shaded region is divided into small squares.

What fraction of the area of the large square is shaded?

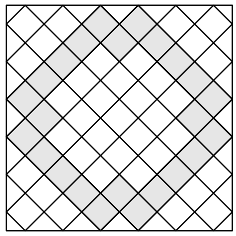
A $\frac{3}{10}$ B $\frac{1}{3}$ C $\frac{3}{8}$ D $\frac{2}{5}$ E $\frac{3}{7}$



Solution **B**

Method 1

We form a complete grid inside the large square by extending the sides of the small squares, as shown in the figure.



In this way the large square is divided up into 61 small squares, 20 triangles along the edges whose areas are equal to half that of the small squares, and 4 triangles in the corners whose areas are equal to one quarter that of the small squares.

So the area of the large square is equal to that of

$$61 \times 1 + 20 \times \frac{1}{2} + 4 \times \frac{1}{4} = 72$$

of the small squares. The shaded area is made up of 24 small squares. Therefore the fraction of the large square that is shaded is

$$\frac{24}{72} = \frac{1}{3}.$$

[Note that we could save some work by taking advantage of the symmetry of the figure, and just work out which fraction of, say, the top left hand corner of the square is shaded.]

Method 2

We let s be the side length of the large square, and t be the side length of the small squares.

It follows that the small squares have diagonals of length $t\sqrt{2}$. [See Problem 19.3, above.]

It can be seen from the diagram that the side length of the large square is equal to the total length of the diagonals of 6 of the small squares. That is,

$$s = 6 \times t\sqrt{2},$$

and therefore

$$t = \frac{s}{6\sqrt{2}}.$$

It follows that

$$t^2 = \frac{s^2}{72}.$$

Hence the total area of the 24 shaded squares is

$$24 \times t^2 = 24 \times \frac{s^2}{72} = \frac{1}{3}s^2.$$

Therefore the fraction of the area of the large square that is shaded is $\frac{1}{3}$.

23. There are 120 different ways of arranging the letters, U, K, M, I and C. All of these arrangements are listed in dictionary order, starting with CIKMU.

Which position in the list does UKIMC occupy?

A 110th B 112th C 114th D 116th E 118th

Solution **B**

Method 1

Because the sequence UKIMC comes towards the bottom of the list in dictionary order of all 120 arrangements of the letters U, K, M, I, C, we can most easily find its position by listing the arrangements in reverse dictionary order, until we reach UKIMC, as follows

120	UMKIC
119	UMKCI
118	UMIKC
117	UMICK
116	UMCKI
115	UMCIK
114	UKMIC
113	UKMCI
112	UKIMC

We see from this that UKIMC is the 112th arrangement in the dictionary order list.

Method 2

It would take rather too long to list all the arrangements in dictionary order from the start, but we can find the position of UKMIC by counting.

Of the 120 arrangements of the letters U, K, M, I, C, there are 24 beginning with each of the 5 letters. It follows that, in dictionary order, the first $24 \times 4 = 96$ of these arrangements begin with C, I, K or M. After these comes the first arrangement, UCIKM, beginning with U. This arrangement therefore occurs in position 97.

Of the 24 arrangements beginning with U there are 6 beginning with each of the remaining 4 letters. Hence, starting from position 97, there are, first, 6 arrangements beginning with UC, and next, 6 beginning UI, before the first arrangement, UKCIM, which begins UK. This arrangement is therefore in position $97 + 12 = 109$.

Of the 6 arrangements beginning UK, there are 2 beginning with each of the remaining 3 letters. The first 2 of these, in positions 109 and 110, are UKCMI and UKCIM. Next there are the two arrangements beginning UKI. These are UKICM and UKIMC. There are in positions 111 and 112, respectively. Therefore UKIMC occurs in position 112.

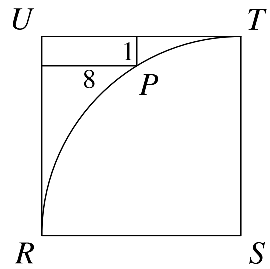
For investigation

23.1 If the 120 arrangements of the letters U, S, M, I, C are put in dictionary order, in which position is the arrangement MUSIC?

24. In square $RSTU$ a quarter-circle arc with centre S is drawn from T to R. A point P on this arc is 1 unit from TU and 8 units from RU.

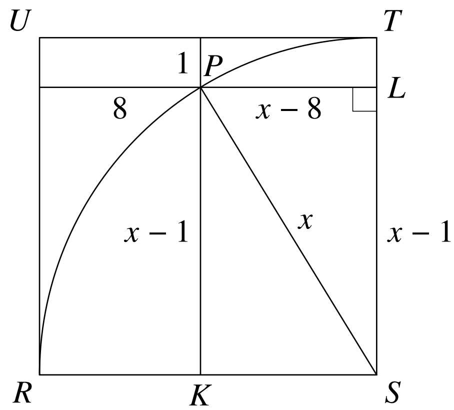
What is the length of the side of square $RSTU$?

A 9 B 10 C 11 D 12 E 13



SOLUTION **E**

Suppose that x is the length of the sides of the square $RSTU$. We need to find the value of x.



We let K be the point where the perpendicular from P to RS meets RS, and L be the point where the perpendicular from P to ST meets ST, as shown in the figure. Then PK has length $x - 1$, and so also does LS. Also, PL has length $x - 8$.

Because PS is a radius of the quarter circle, its length is the same as that of the sides of the square, namely, x.

Therefore SLP is a right-angled triangle in which the hypotenuse has length x, and the other two sides have lengths $x - 1$ and $x - 8$. Hence, by Pythagoras' Theorem,

$$(x-1)^2 + (x-8)^2 = x^2.$$

Multiplying out the squares, we get

$$(x^2 - 2x + 1) + (x^2 - 16x + 64) = x^2,$$

which we can rearrange as

$$x^2 - 18x + 65 = 0.$$

We can factorize the quadratic on the left-hand side of this equation to give the equivalent equation

$$(x-5)(x-13) = 0,$$

which has the solutions $x = 5$ and $x = 13$. Since $x - 8$ is the length of PL and therefore cannot be negative, $x \neq 5$. Therefore $x = 13$.

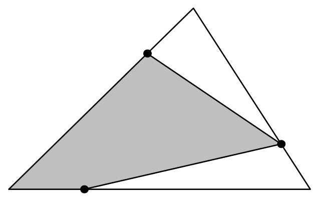
FOR INVESTIGATION

24.1 How can we interpret the solution $x = 5$ of the quadratic equation which occurs in the solution above?

25. A point is marked one quarter of the way along each side of a triangle, as shown.

What fraction of the area of the triangle is shaded?

A $\frac{7}{16}$ B $\frac{1}{2}$ C $\frac{9}{16}$ D $\frac{5}{8}$ E $\frac{11}{16}$

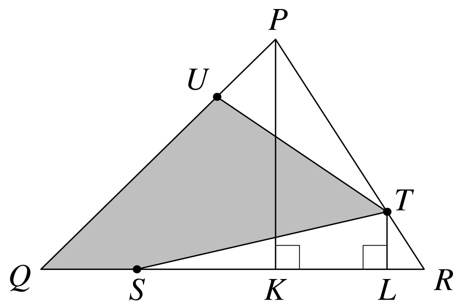


SOLUTION **D**

We give two methods for solving this problem. We use the notation $\triangle XYZ$ for the area of a triangle XYZ.

METHOD 1

We let P, Q and R be the vertices of the triangle, and let S, T and U be the points one quarter of the way along each side, as shown in the figure.



We let K and L be the points where the perpendiculars from P and T, respectively, meet QR.

The area of a triangle is half the base multiplied by the height of the triangle. Therefore

$$\triangle PQR = \tfrac{1}{2}(QR \times PK).$$

We now obtain an expression for $\triangle TSR$.

Because S is one quarter of the way along QR, $SR = \frac{3}{4}QR$.

In the triangles PKR and TLR, the angles at K and L are both right angles, and the angle at R is common to both triangles. It follows that these triangles are similar. Therefore

$$\frac{TL}{PK} = \frac{TR}{PR} = \frac{1}{4},$$

and hence

$$TL = \tfrac{1}{4}PK.$$

It follows that

$$\triangle STR = \tfrac{1}{2}(SR \times TL) = \tfrac{1}{2}(\tfrac{3}{4}QR \times \tfrac{1}{4}PK) = \tfrac{3}{16}\left(\tfrac{1}{2}(QR \times PK)\right) = \tfrac{3}{16}\triangle PQR.$$

In Problem 25.1 you are asked to show that, similarly,

$$\triangle TPU = \tfrac{3}{16}\triangle PQR.$$

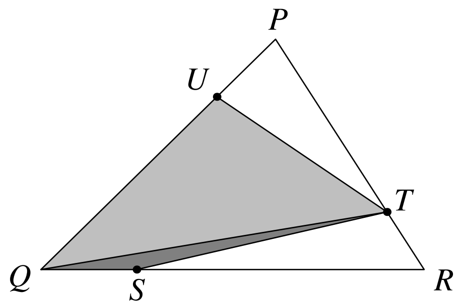
It follows that the fraction of the area of triangle PQR that is shaded is

$$1 - \frac{3}{16} - \frac{3}{16} = \frac{5}{8}.$$

Method 2

This, more elegant, approach uses the result that *the ratio of the areas of two triangles with the same height is equal to the ratio of their bases.* If you haven't met this before, you are encouraged to find a proof (see Problem 25.2).

We join Q and T as shown in the figure.



The triangles QTS and QTR have the same height. Therefore, using the above result,

$$\triangle QTS : \triangle QTR = QS : QR = 1 : 4.$$

In other words

$$\triangle QTS = \tfrac{1}{4}\triangle QTR.$$

The triangles QTR and PQR have the same height, and therefore

$$\triangle QTR : \triangle PQR = TR : PR = 1 : 4.$$

In other words

$$\triangle QTR = \tfrac{1}{4}\triangle PQR.$$

Therefore

$$\triangle QTS = \tfrac{1}{4}\triangle QTR = \tfrac{1}{4}(\tfrac{1}{4}\triangle PQR) = \tfrac{1}{16}\triangle PQR.$$

Similarly (see Problem 25.3),

$$\triangle QTU = \tfrac{9}{16}\triangle PQR.$$

The shaded region is made up of the triangles QTS and QTU. Therefore its area is

$$\begin{aligned} \triangle QTS + \triangle QTU &= \tfrac{1}{16}\triangle PQR + \tfrac{9}{16}\triangle PQR \\ &= \left(\tfrac{1}{16} + \tfrac{9}{16}\right)\triangle PQR \\ &= \tfrac{10}{16}\triangle PQR \\ &= \tfrac{5}{8}\triangle PQR. \end{aligned}$$

For investigation

25.1 Show that, as used in Method 1, $\triangle TPU = \frac{3}{16}\triangle PQR$.

25.2 Prove that the ratio of the areas of two triangles with the same height is equal to the ratio of their bases.

25.3 Verify that, as used in Method 2,

$$\triangle QTU = \tfrac{9}{16}\triangle PQR.$$